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A Refinement of an Inequality of S. Bernstein

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Let $P(z)$ be a polynomial of degree n and $P'(z)$ its derivative. Using a recently developed interpolation formula, we obtain certain interesting refinements of the well-known inequalities of S. Bernstein and M. Riesz for polynomials. Given $P(1)=0$, the problem of estimating $|P(r)|$, with $0 \leqslant r < 1$, is also taken up. Finally we present a sharp lower bound concerning the maximum of $|P'(z)|$ on $|z|=1$.

1. INTRODUCTION AND STATEMENT OF RESULTS

If $P(z)$ is a polynomial of degree n, then concerning the estimate of the maximum of $|P'(z)|$ on the unit circle $|z|=1$ and the estimate of the maximum of $|P(z)|$ on a larger circle $|z|=R>1$, we have

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant n \operatorname*{Max}_{|z|=1} |P(z)| \tag{1}$$

and

$$\operatorname*{Max}_{|z|=R>1} |P(z)| \leqslant R^n \operatorname*{Max}_{|z|=1} |P(z)|. \tag{2}$$

Inequality (1) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for reference see [8]). Inequality (2) is a simple deduction from the maximum modulus principle (see [7, p. 346] or [6, Vol. I, p. 137]).

In both (1) and (2) equality holds only for $P(z)=\alpha z^n$, $|\alpha| \neq 0$, that is, if and only if $P(z)$ has all its zeros at the origin. Recently it was shown by Frappier, Rahman, and Ruscheweyh [3, Theorem 8] that if $P(z)$ is a polynomial of degree n, then

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant n \operatorname*{Max}_{1 \leqslant k \leqslant 2n} |P(e^{ik\pi/n})|. \tag{3}$$

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Clearly (3) represents a refinement of (1), since the maximum of $|P(z)|$ on $|z|=1$ may be larger than the maximum of $|P(z)|$ taken over the $(2n)$th roots of unity, as is shown by the simple example $P(z)=z^n+ia$, $a>0$. In this paper we shall first show that the bound in (3) can be considerably improved. In fact, we prove the following result which constitutes an interesting refinement of (3) and therefore of Bernstein's inequality (1) as well.

THEOREM 1. *If $P(z)$ is a polynomial of degree n, then for every given real α,*

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{n}{2}(M_\alpha + M_{\alpha+\pi}), \tag{4}$$

where

$$M_\alpha = \operatorname*{Max}_{1\leqslant k\leqslant n} |P(e^{i(\alpha+2k\pi)/n})| \tag{5}$$

and $M_{\alpha+\pi}$ is obtained from (5) by replacing α by $\alpha+\pi$. The result is the best possible and equality in (4) holds for $P(z)=z^n+re^{i\alpha}$, $-1\leqslant r\leqslant 1$.

Taking $\alpha=0$ in Theorem 1, we obtain

COROLLARY 1. *If $P(z)$ is a polynomial of degree n, then*

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{n}{2}\{\operatorname*{Max}_{1\leqslant k\leqslant n} |P(e^{2k\pi i/n})| + \operatorname*{Max}_{1\leqslant k\leqslant n} |P(e^{(1+2k)\pi i/n})|\}; \tag{6}$$

that is, on the right hand side of (3), the maximum of $|P(z)|$ taken over the $(2n)$th roots of unity may be replaced by the arithmetic mean of the maximum of $|P(z)|$ taken over the nth roots of 1 and the maximum of $|P(z)|$ taken over the nth roots of -1. The result is the best possible and equality in (6) holds for $P(z)=z^n-r$, $-1\leqslant r\leqslant 1$.

As an application of Theorem 1, we next present the following result which constitutes the corresponding refinement of (2).

THEOREM 2. *If $P(z)$ is a polynomial of degree n, then for all real α and $R>1$,*

$$\operatorname*{Max}_{|z|=1} |P(Rz)-P(z)| \leqslant \left(\frac{R^n-1}{2}\right)(M_\alpha + M_{\alpha+\pi}), \tag{7}$$

where M_α is defined by (5) and $M_{\alpha+\pi}$ is obtained from M_α by replacing α

by $\alpha+\pi$. *The result is the best possible and equality in* (7) *holds for the polynomial* $P(z)=z^n+re^{i\alpha}$, $-1\leqslant r\leqslant 1$.

As an immediate consequence of Theorem 2, we obtain

COROLLARY 2. *If* $P(z)$ *is a polynomial of degree* n, *then for all* $R>1$,

$$\operatorname*{Max}_{|z|=R>1}|P(z)|\leqslant\left(\frac{R^n-1}{2}\right)(M_0+M_\pi)+\operatorname*{Max}_{|z|=1}|P(z)|, \tag{8}$$

where M_α *is defined by* (5) *for all real* α. *The result is the best possible with equality in* (8) *for* $P(z)=z^n-r$, $-1\leqslant r\leqslant 1$.

Here is another consequence of Theorem 2, which is obtained by applying (7) to the polynomial $Q(z)=z^n\overline{P(1/\bar{z})}$ and noting that $|P(z)|=|Q(z)|$ for $|z|=1$.

COROLLARY 3. *If* $P(z)$ *is a polynomial of degree* n, *then for all real* α *and* $r\leqslant 1$,

$$\operatorname*{Max}_{|z|=1}|P(rz)-r^nP(z)|\leqslant\left(\frac{1-r^n}{2}\right)(M_\alpha+M_{\alpha+\pi}) \tag{9}$$

and a fortiori

$$\operatorname*{Max}_{|z|=r<1}|P(z)|\leqslant\left(\frac{1-r^n}{2}\right)(M_0+M_\pi)+r^n\operatorname*{Max}_{|z|=1}|P(z)|, \tag{10}$$

where M_α *is defined by* (5) *for all* α. *Both the estimates are sharp, with equality in* (9) *for* $P(z)=az^n+e^{i\alpha}$, $-1\leqslant a\leqslant 1$ *and in* (10) *for* $P(z)=z^n+1$.

Several years ago, in quite a different context G. Halász of the Mathematical Institute of the Hungarian Academy of Sciences asked how large $\operatorname{Min}_{|z|=1-(w/n)}|P(z)|$ can be for a given w in $(0,n]$ if $P(1)=0$? As an answer to this question, Giroux and Rahman [4, Theorem 6] have shown that if $P(z)$ is a polynomial of degree n such that $P(1)=0$, then for $0<w\leqslant n$,

$$\left|P\left(1-\frac{w}{n}\right)\right|\leqslant\left\{1-\left(\frac{1}{w}-\frac{1}{2n}\right)+\left(\frac{1}{w}-\frac{1}{2n}\right)\left(1-\frac{w}{n}\right)^n\right\}\times\operatorname*{Max}_{1\leqslant k\leqslant 2n-1}|P(e^{ik\pi/n})|. \tag{11}$$

Here we establish the following result which gives a sharp estimate for $\operatorname{Min}_{|z|=1-(w/n)}|P(z)|$, $0\leqslant w\leqslant n$.

THEOREM 3. *If $P(z)$ is a polynomial of degree n such that $P(1)=0$, then for $0 \leqslant w \leqslant n$,*

$$\left|P\left(1-\frac{w}{n}\right)\right| \leqslant \frac{1}{2}\left\{1-\left(1-\frac{w}{n}\right)^{n}\right\}(M_0+M_\pi), \tag{12}$$

where M_α is defined by (5). The result is the best possible and equality in (12) holds for $P(z)=z^n-1$.

Inequality (1) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z|<1$. In fact, P. Erdös conjectured and later P. D. Lax [5] (see also [2]) verified that if $P(z) \neq 0$ in $|z|<1$, then (1) can be replaced by

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \operatorname*{Max}_{|z|=1} |P(z)|. \tag{13}$$

In this connection it is natural to ask what improvement results in (4) from supposing that $P(z) \neq 0$ in $|z|<1$. Here we are able to prove

THEOREM 4. *If $P(z)$ is a polynomial of degree n having no zeros in $|z|<1$, then for every given real α,*

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{n}{2}\{M_\alpha^2+M_{\alpha+\pi}^2\}^{1/2}, \tag{14}$$

where M_α is defined by (5) for all real α. The result is the best possible and equality in (14) holds for $P(z)=z^n+e^{i\alpha}$.

As an application of Theorem 4, we mention the corresponding improvement of (7).

THEOREM 5. *If $P(z)$ is a polynomial of degree n having no zeros in the disk $|z|<1$, then for every given real α and $R>1$,*

$$\operatorname*{Max}_{|z|=1} |P(Rz)-P(z)| \leqslant \left(\frac{R^n-1}{2}\right)(M_\alpha^2+M_{\alpha+\pi}^2)^{1/2}, \tag{15}$$

where M_α is defined by (5). The result is sharp with equality in (15) for $P(z)=z^n+e^{i\alpha}$.

Finally we present the following result concerning a lower bound for the maximum of $|P'(z)|$ on $|z|=1$.

THEOREM 6. *If $P(z)$ is a polynomial of degree n which has m zeros at the origin, $0 \leqslant m \leqslant n$, then*

$$\operatorname*{Max}_{|z|=1} |P'(z)| \geqslant \frac{n}{2}\left(2 \operatorname*{Max}_{|z|=1} |P(z)| - \left(1 - \frac{m}{n}\right)(M_1^* + M_2^*)\right), \tag{16}$$

where

$$M_1^* = \operatorname*{Max}_{1 \leqslant k \leqslant n-m} |P(e^{2k\pi i/(n-m)})|$$

and

$$M_2^* = \operatorname*{Max}_{1 \leqslant k \leqslant n-m} |P(e^{(1+2k)\pi i/(n-m)})|.$$

The result is the best possible and equality in (16) *holds for $P(z) = z^n + rz^m$, for every $r \geqslant 1$.*

The case $m = 0$ yields the following

COROLLARY 4. *If $P(z)$ is a polynomial of degree n, then*

$$\operatorname*{Max}_{|z|=1} |P'(z)| \geqslant \frac{n}{2}(2 \operatorname*{Max}_{|z|=1} |P(z)| - (M_0 + M_\pi)), \tag{17}$$

where M_α is defined by (5) *for all real α. The result is the best possible with equlity in* (17) *for $P(z) = z^n + r$, for every $r \geqslant 1$.*

2. LEMMAS

For the proofs of these theorems we need the following lemmas. The first result is an interpolation formula due to the authore [1].

LEMMA 1. *If $P(z)$ is a polynomial of degree n and $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + a$, where a is any non-zero complex number, then for every complex number t such that $t^n + a \neq 0$, we have*

$$P'(t) = \frac{nt^{n-1}}{a+t^n} P(t) + \frac{a+t^n}{na} \sum_{k=1}^{n} P(z_k) \frac{z_k}{(z_k - t)^2} \tag{18}$$

and

$$\frac{1}{na} \sum_{k=1}^{n} \frac{z_k t}{(z_k - t)^2} = -\frac{nt^n}{(a+t^n)^2}. \tag{19}$$

From Lemma 1, we deduce the following

LEMMA 2. *If $P(z)$ is a polynomial of degree n and $z_1, z_2, ..., z_n$ are the zeros of $z^n + e^{i\beta}$, where β is any real number, then for $|z| = 1$,*

$$|z^{n-1}(zP'(z) - nP(z)) + e^{i\beta}P'(z)| \leqslant n \max_{1 \leqslant k \leqslant n} |P(z_k)|. \tag{20}$$

Proof of Lemma 2. In Lemma 1 we take $a = e^{i\beta}$, where β is an arbitrary real number; then the zeros z_k, $k = 1, 2, ..., n$ of $z^n + a$ are of unit modulus. Hence for every complex number t with $|t| = 1$ and $t^n + a \neq 0$, so that $t \neq z_k$, $k = 1, 2, ..., n$, it follows from (18) that

$$\begin{aligned} &|(t^n + a)\, P'(t) - nt^{n-1}P(t)| \\ &= \left| \frac{(a + t^n)^2}{na} \sum_{k=1}^{n} P(z_k) \frac{z_k}{(z_k - t)^2} \right| \\ &\leqslant \left| \frac{(a + t^n)^2}{na} \right| \sum_{k=1}^{n} \left| \frac{z_k}{(z_k - t)^2} \right| \max_{1 \leqslant k \leqslant n} |P(z_k)|. \end{aligned} \tag{21}$$

Now if $|t| = 1$, $|z| = 1$, and $t \neq z$, then it can be easily verified that $zt/(z - t)^2$ is a negative real number and moreover for $|a| = 1$, $|t| = 1$, and $t^n + a \neq 0$, $(a + t^n)^2/at^n$ is a positive real number. Therefore, for $|t| = 1$, $|a| = 1$, and $t^n + a \neq 0$, we have, by (19),

$$\begin{aligned} \left| \frac{(a + t^n)^2}{na} \right| \sum_{k=1}^{n} \left| \frac{z_k}{(z_k - t)^2} \right| &= \left| \frac{(a + t^n)^2}{nat^n} \right| \sum_{k=1}^{n} \left| \frac{tz_k}{(z_k - t)^2} \right| \\ &= -\frac{(a + t^n)^2}{nat^n} \sum_{k=1}^{n} \frac{tz_k}{(z_k - t)^2} = n. \end{aligned}$$

Using this in (21), we obtain

$$|(a + t^n)\, P'(t) - nt^{n-1}P(t)| \leqslant n \max_{1 \leqslant k \leqslant n} |P(z_k)| \tag{22}$$

for $|t| = 1$, $|a| = 1$, and $a + t^n \neq 0$. Since the inequality (22) obviously holds for $t = z_k$, $k = 1, 2, ..., n$ also, it follows that for every real β,

$$|t^{n-1}(tP'(t) - nP(t)) + e^{i\beta}P'(t)| \leqslant n \max_{1 \leqslant k \leqslant n} |P(z_k)|$$

for $|t| = 1$, which is (20), and this completes the proof of Lemma 2.

For the proof of Theorem 4, we need

LEMMA 3. *If $P(z)$ is a polynomial of degree n, then for $|z|=1$ and for every real α,*

$$|P'(z)|^2+|nP(z)-zP'(z)|^2\leqslant\frac{n^2}{2}(M_\alpha^2+M_{\alpha+\pi}^2), \tag{23}$$

where M_α is defined by (5).

Proof of Lemma 3. If $z_1, z_2, ..., z_n$ are the zeros of $z^n+e^{i\beta}$, where β is a real number, then

$$z_k=e^{i(\beta+(1+2k)\pi)/n}, \qquad k=1, 2, ..., n.$$

In Lemma 2 we take first $\beta=\alpha$ and next $\beta=\alpha-\pi$, where α is any given real number, and obtain, for $|z|=1$,

$$\begin{aligned}&|z^{n-1}(zP'(z)-nP(z))+e^{i\alpha}P'(z)|\\ &\qquad\leqslant n\max_{1\leqslant k\leqslant n}|P(e^{i(\alpha+(1+2k)\pi)/n}|=nM_{\alpha+\pi}\end{aligned} \tag{24}$$

and

$$\begin{aligned}&|z^{n-1}(zP'(z)-nP(z))-e^{i\alpha}P'(z)|\\ &\qquad\leqslant n\max_{1\leqslant k\leqslant n}|P(e^{i(\alpha+2k\pi)/n}|=nM_\alpha.\end{aligned} \tag{25}$$

From (24) and (25), it follows that

$$\begin{aligned}&|z^{n-1}(zP'(z)-nP(z))+e^{i\alpha}P'(z)|^2\\ &\qquad+|z^{n-1}(zP'(z)-nP(z))-e^{i\alpha}P'(z)|^2\\ &\leqslant n^2(M_\alpha^2+M_{\alpha+\pi}^2) \qquad \text{for} \quad |z|=1.\end{aligned} \tag{26}$$

Using now the identity

$$|A+B|^2+|A-B|^2=2|A|^2+2|B|^2$$

in (26) with

$$A=z^{n-1}(zP'(z)-nP(z)) \qquad \text{and} \qquad B=e^{i\alpha}P'(z),$$

we get, for $|z|=1$,

$$2(|zP'(z)-nP(z)|^2+|P'(z)|^2)\leqslant n^2(M_\alpha^2+M_{\alpha+\pi}^2),$$

which is equivalent to (23), and Lemma 3 is proved.

3. Proofs of the Theorems

Proof of Theorem 1. If we proceed as in the proof of Lemma 3, it follows from (24) and (25) that for every given real number α and for $|z|=1$,

$$\begin{aligned} 2\,|P'(z)| &= 2\,|e^{i\alpha}P'(z)| \\ &\leqslant |e^{i\alpha}P'(z) - z^{n-1}(nP(z) - zP'(z))| \\ &\quad + |e^{i\alpha}P'(z) + z^{n-1}(nP(z) - zP'(z))| \\ &\leqslant n(M_{\alpha+\pi} + M_\alpha). \end{aligned}$$

This implies

$$\operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{n}{2}(M_\alpha + M_{\alpha+\pi}),$$

which is the desired result.

Proof of Theorem 2. Applying (2) to the polynomial $P'(z)$, which is of degree $n-1$, and using Theorem 1, we obtain for all $t \geqslant 1$ and $0 \leqslant \theta < 2\pi$

$$|P'(te^{i\theta})| \leqslant t^{n-1} \operatorname*{Max}_{|z|=1} |P'(z)| \leqslant \frac{nt^{n-1}}{2}(M_\alpha + M_{\alpha+\pi}).$$

Hence for each θ, $0 \leqslant \theta < 2\pi$ and $R > 1$, we have

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| &= \left| \int_1^R e^{i\theta} P'(te^{i\theta})\, dt \right| \\ &\leqslant \int_1^R |P'(te^{i\theta})|\, dt \\ &\leqslant \frac{1}{2}(M_\alpha + M_{\alpha+\pi}) \int_1^R nt^{n-1}\, dt \\ &= \frac{(R^n - 1)}{2}(M_\alpha + M_{\alpha+\pi}). \end{aligned}$$

This implies

$$|P(Rz) - P(z)| \leqslant \frac{(R^n - 1)}{2}(M_\alpha + M_{\alpha+\pi}) \qquad \text{for} \quad |z| = 1 \text{ and } R > 1,$$

which is equivalent to the desired result.

Proof of Theorem 3. If $Q(z)=z^n\overline{P(1/\bar{z})}$, then $|Q(z)|=|P(z)|$ for $|z|=1$ and by the hypothesis of Theorem 3 we have $Q(1)=\overline{P(1)}=0$. Applying Theorem 2 to the polynomial $Q(z)$ with $\alpha=0$, we get, for $R>1$,

$$|Q(R)|\leqslant\frac{(R^n-1)}{2}(M_0+M_\pi).$$

This implies, for $R>1$,

$$|P(1/R)|\leqslant\tfrac{1}{2}(1-R^{-n})(M_0+M_\pi).$$

If $0<w\leqslant n$, then $(1-w/n)^{-1}>1$ and therefore, in particular, we have

$$\left|P\left(1-\frac{w}{n}\right)\right|\leqslant\frac{1}{2}\left(1-\left(1-\frac{w}{n}\right)^n\right)(M_0+M_\pi),$$

and hence the proof of Theorem 3 is complete.

Proof of Theorem 4. Since the polynomial $P(z)$ does not vanish in the disk $|z|<1$, we note [2, p. 121] that

$$|P'(z)|\leqslant|nP(z)-zP'(z)|\qquad\text{for}\quad|z|=1. \tag{27}$$

Combining the inequality (27) with the conclusion of Lemma 3, we obtain

$$\begin{aligned}2\,|P'(z)|^2&\leqslant|P'(z)|^2+|nP(z)-zP'(z)|^2\\&\leqslant\frac{n^2}{2}(M_\alpha^2+M_{\alpha+\pi}^2)\qquad\text{for}\quad|z|=1,\end{aligned}$$

from which Theorem 4 follows immediately.

Proof of Theorem 5. This theorem follows easily on using arguments similar to that used in the proof of Theorem 2, and therefore we omit the details.

Proof of Theorem 6. Since the polynomial $P(z)$ has m zeros at the origin, $0\leqslant m\leqslant n$, we write $P(z)=z^mH(z)$, where $H(z)$ is a polynomial of degree $n-m$ and $H(0)\neq 0$. If $Q(z)=z^n\overline{P(1/\bar{z})}$, then clearly $Q(z)$ is a polynomial of degree $n-m$ and $Q(z)=z^{n-m}\overline{H(1/\bar{z})}$. Also it is immediate that

$$|Q(z)|=|H(z)|=|P(z)|\qquad\text{for}\quad|z|=1. \tag{28}$$

Moreover,

$$zQ'(z)=nz^n\overline{P(1/\bar{z})}-z^{n-1}\overline{P'(1/\bar{z})},$$

from which it follows that for $|z|=1$,

$$|Q'(z)|=|z^{n-1}\overline{Q'(1/\bar{z})}|=|nP(z)-zP'(z)|. \tag{29}$$

Applying Corollary 1 to the polynomial $Q(z)$ and keeping in mind that by (28),

$$\max_{1 \leqslant k \leqslant n-m} |Q(e^{2k\pi i/(n-m)})| = \max_{1 \leqslant k \leqslant n-m} |P(e^{2k\pi i/(n-m)})| = M_1^*,$$

$$\max_{1 \leqslant k \leqslant n-m} |Q(e^{(1+2k)\pi i/(n-m)})| = \max_{1 \leqslant k \leqslant n-m} |P(e^{(1+2k)\pi i/(n-m)})| = M_2^*,$$

we obtain

$$|Q'(z)| \leqslant \frac{(n-m)}{2}(M_1^* + M_2^*) \qquad \text{for} \quad |z| = 1.$$

This implies with the help of (29) that

$$|nP(z) - zP'(z) \leqslant \frac{(n-m)}{2}(M_1^* + M_2^*) \qquad \text{for} \quad |z| = 1,$$

which further implies that

$$n\,|P(z)| \leqslant \frac{(n-m)}{2}(M_1^* + M_2^*) + |P'(z)| \qquad \text{for} \quad |z| = 1.$$

This gives

$$n \max_{|z|=1} |P(z)| \leqslant \frac{(n-m)}{2}(M_1^* + M_2^*) + \max_{|z|=1} |P'(z)|,$$

from which the desired result follows immediately.

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